16MSPCH302 — NUCLEAR AND PARTICLE PHYSICS

THE FERMI GAS MODEL OF NUCLEI

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1 INTRODUCTION

NUCLEAR MODELS ARE an excellent tool to simplify and approximate nuclear structures while keeping (most) core properties of nuclei intact. During previous lectures we spent time discussing the liquid drop and shell models. Recall that both of these required nucleon interactions. There is an alternate model called the Fermi gas model that describes nuclei as an assortment of nucleons, each of which behaves completely independently of the others.

Both protons and neutrons are fermions, particles with half-integral spin and which follow Fermi-Dirac statistics. The Fermi gas model puts all nucleons inside an infinite potential well with no explicit requirement that they interact with one another. Consequently the description provided by the Fermi gas model is one that looks at the overall nature of the nucleus rather than the individual behaviour of its constituent particles.

The idea of two separate energy wells for protons and neutrons is not new to us: we discussed a similar set-up as part of the Bethe-Weiszäcker semi-empirical mass formula.

Imagine that the protons and neutrons in a nucleus occupy their own potential wells with each nucleon occupying the lowest possible energy state. They fill up the well until they reach the Fermi energy level (say E_F). If they have a common E_F they are stable, else the excess nucleons interconvert[1] to achieve such stability.

2 INFINITE POTENTIAL WELL

We discuss a three-dimensional well here. You can also discuss a two-dimensional case which turns out to be much simpler but also somewhat less accurate. A third alternative is to simplify the problem using a cube and Cartesian coördinates instead of a sphere and polar coördinates [1].

Say our well has dimensions of L_x, L_y and L_z along the x, y and z directions. This gives us the following wavefunctions:

$$\begin{aligned}\psi_{n_x,n_y,n_z}(x,y,z) &= \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)\\ &= \sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right)\end{aligned}$$

using the usual relation $k_i L_i = n_i\pi$ for well dimension L_i in the i direction. Combining

We use the $E\psi = -\hbar^2\nabla^2\psi/2m$ form of Schrödinger's equation here. And $k_i L_i = n_i\pi$ comes from the solution to the standard 'particle in a box' problem.

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[1] β decay.

this with Schrödinger's equation we get the energies

$$E_{n_i} = \frac{\pi^2 \hbar^2 n_i^2}{2mL_i^2}$$

Note also that in eq. (1) the energy is equal to $\sum p_i^2/2m$ as a result of E being $p^2/2m$ for components $p_i = \hbar k_i = \hbar \pi n_i / L_i$ of the momenta of a particle.

which consists of three terms as per the summation convention. We can extend this to an $L_x = L_y = L_z = L$ type well to get a simpler formula:

$$E_{n_i} = \frac{\pi^2 \hbar^2}{2mL^2} \left(n_i^2 \right) \tag{1}$$

Keep in mind, though, that such a square well is only an ideal case. Equation (1) gives us part of the depth of the potential well. This, plus the binding energy of the outermost nucleons (≈ 8 MeV empirically) gives us the total depth. Our next exercise, therefore, is to try to determine the Fermi energy.

3 FERMI MOMENTUM AND ENERGY

The Fermi momentum p_F is clearly given by $\sqrt{2mE_F}$ (see margin note above), and the momenta of nucleons, in each direction, is less than p_F in that $\sum p_i^2 = \pi^2 \hbar^2 n_i^2 / L^2 < p_F^2$. In other words,

$$\sum_i n_i^2 < \frac{p_F^2 L^2}{\pi^2 \hbar^2} = R^2 \text{ (say)} \tag{2}$$

The fact that each state of our system is described by an (n_x, n_y, n_z) set of quantum numbers means each change in this set (each n_i changes after another) describes a new state of our system. That is, each change in n_i is a new unit change in volume and a new state; we therefore have a system with one state per unit volume. This is an important observation.

Since all x_i exist – if they go to zero E_F disappears – we need only discuss $(1/2^n)^{th}$ of the sphere without sacrificing accuracy; use $n = 3$ dimensions here to get $1/8$.

The condition described by inequality (2) is a sphere of radius $R = p_F L / \pi \hbar$. The volume of this sphere ($4\pi R^3/3$) then gives us the number of states up to p_F. Considering an eighth of this sphere and the fact that 2 types of spins exist for each nucleon and we have

$$\begin{aligned} n &= \frac{2}{8} \frac{4}{3} \pi R^3 \\ &= \frac{\pi}{3} \left(\frac{p_F L}{\pi \hbar} \right)^3 \\ \Rightarrow p_F &= \hbar \sqrt[3]{3\pi^2} \sqrt[3]{\frac{n}{L^3}} \end{aligned} \tag{3}$$

where $L^3 = V$ is the volume of our (assumed) cubic well.

Equation (3) gives us the Fermi momentum in terms of the density of states n/V which is all well and good, but what is it in terms of Z and A, the quantities we have been most interested in so far? We know that $R = R_0 \sqrt[3]{A}$ which gives us a volume $V = 4\pi R_0 \sqrt[3]{A}/3$ which can be plugged into eq. (3) to get

Equation (4) is valid only for protons. For neutrons $Z \to (A - Z)$ inside the cube root.

$$p_F = \frac{\hbar}{R_0} \sqrt[3]{\frac{9\pi}{4} \frac{Z}{A}} \tag{4}$$

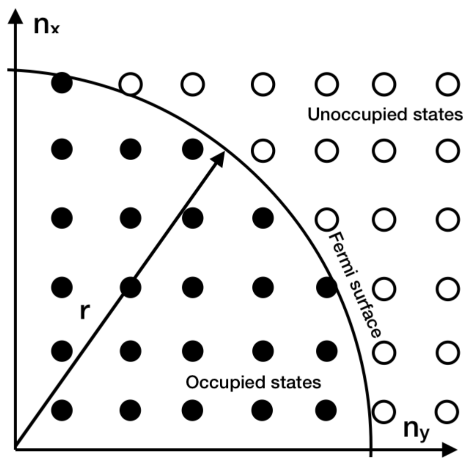

Figure 1: Fermi sphere.

The Fermi energy of a nucleus follows logically from the expression for Fermi momentum given above. An important point to keep in mind though, again, harking back to Weiszäcker's formula, is that the proton and neutron wells may be filled differently. This results in different values of E_F for protons and neutrons. For protons

$$E_F = \frac{\hbar^2}{2R_0^2 m_p}\left(\frac{9\pi}{4}\frac{Z}{A}\right)^{2/3} \tag{5}$$

and for neutrons $Z \rightarrow (A - Z)$ in the numerator on the right-hand side. It is easy to see that when the number of protons and neutrons are equal i.e. $A = 2Z$ the Fermi energy becomes $\approx 33\,\text{MeV}$ (considering $R_0 = 1.2\,\text{fm}$). This plus the 8 MeV binding energy of the outer nucleons stated earlier gives us a potential well (for both protons and neutrons) that is approximately 41 MeV deep. Needless to say this number varies by element. Further, it need not always be equal for both protons and neutrons but, as in our example, it sometimes might be so.

The reason why protons and neutrons have different well depths is also partly due to the Coulomb repulsion between protons that is absent from neutrons. Yet again, this is an idea we already encountered in our discussions of the SEMF. Observe in fig. 2 that a difference in well depths implies a difference in the Fermi energy levels. That is to say deeper potential wells imply a higher maximum occupied energy level. The difference in Fermi levels prompts continuous β decays to convert protons to neutrons and vice versa until the well depths are equalised. Such β decays result in a gain/loss of energy across the two wells. Once equalised the nucleus becomes stable and no more decays are observed. Figure 2 shows an unstable nucleus; after β decays the wells stabilise towards equal Fermi energies of $43\,\text{MeV} > E_F > 37\,\text{MeV}$.

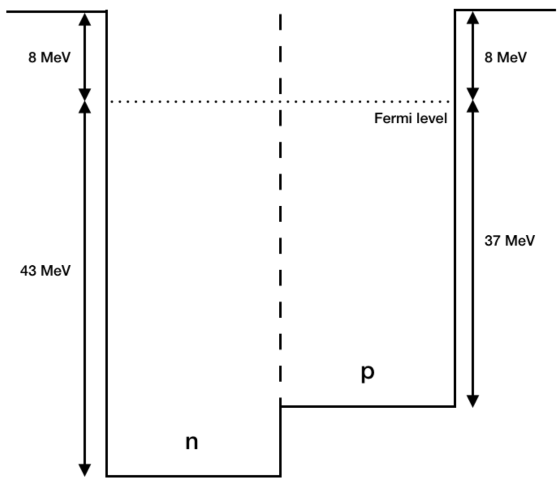

Figure 2: Example well depths.

4 NUCLEAR LEVEL DENSITY

The simplest definition of nuclear level density $\rho(E)$ is given by the number of levels per unit energy. Alternately it gives the number of ways of arranging nucleons in orbitals to achieve an energy between E and some $E + dE$. The level densities of some nuclei are still unknown but, even among known result, the predictions of the Fermi gas model deviate slightly from empirical measurements. Mathematically level density is given by

$$\rho(E) = \frac{dN}{dE_{n_i}}$$

You can just as well consider (r, θ) but we retain n here for consistency in variable representation.

The term N gives us the number of (n_x, n_y, n_z) triplets. We can write these Cartesian coördinates in Polar form using $dn_x + dn_y + dn_z = dN = n^2 dn\, d\Omega$ where n and Ω are the polar coördinates.

What eq. (1) tells us is precisely the value of n in the above expansion. The result of this (see [2], eq. (6.8) for a slightly more detailed explanation) turns out to be

$$dN = dE_{n_i} \frac{L^3}{\pi^2 \sqrt{s}} \left(\frac{m}{\hbar}\right)^{3/2} \sqrt{E_{n_i}}$$

You can try this yourself as an exercise: for the same reasons as those discussed in section 3 above we have $d\Omega \to 4\pi/8$ and any negatives obtained while differentiating eq. (1) can be omitted since only positive values are allowed for n. Using this, and the value of n^2 obtained by rearranging eq. (1), simple substitution gives you the expression for dN stated here.

Since only states up to the Fermi energy are filled we need to integrate this over that region alone. Further, dividing that result by the volume (assumed to be a cube of L^3 here) gives us our level density.

$$N = \frac{L^3}{\pi^2 \sqrt{2}} \left(\frac{m}{\hbar}\right)^{3/2} \int_0^{E_F} dE_{n_i} \sqrt{E_{n_i}}$$

$$\Rightarrow \rho = \frac{1}{\pi^2 \sqrt{2}} \left(\frac{m}{\hbar}\right)^{3/2} \int_0^{E_F} dE_{n_i} \sqrt{E_{n_i}} \tag{6}$$

The Fermi gas model leads to an interesting process called nuclear evaporation [5] that is often compared to fission. Indeed it is quite like fission but at lower energies. At high energies particles bombarding a nucleus have a de Broglie wavelength in the order of femtometres allowing them to interact with individual nucleons.

Within about 10^{-22} s a cascade of reactions takes place between 20 MeV and the energy of the incident particle (usually 100 MeV or above) leading to the creation of various secondary particles. The general result of such bombardment is that nuclei get excited. Following the excitation process is naturally a de-excitation. This is known as **nuclear evaporation** and leads to the the loss/ejection of neutrons and protons from the nucleus. This is distinct from fission where bombardment itself has this effect. Nuclear evaporation usually takes place within about 10^{-16} s and results in particles with energies that are usually below about 20 MeV. Also, whereas nuclear cascades lead to particles ejections with anisotropic energies, evaporation is isotropic.

References

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